AN INTRODUCTION TO STATISTICAL METHODS - 27 October 2006

Assignment 1 - Solutions[1]

Ex. 1 Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

Solution: Let G and C be the events that the chosen student is a genius and a chocolate lover, respectively. We have $P(G) = 0.6$,$P(C) = 0.7$, and $P(G \bigcap C) = 0.4$. We are interested in $P(G^c \cap C^c)$, which is obtained with the following calculation :

$$P(G^c \cap C^c) = 1 - P(G \cup C) = 1 - (P(G) + P(C) - P(G \cap C)) = 1 - (0.6 + 0.7 - 0.4) = 0.1.$$

Ex. 2 We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

(a) Find the probability that doubles were rolled.

(b) Given that the roll resulted in a sum of 4 or less, find the conditional probability that doubles were rolled.

(c) Find the probability that at least one die is a 6.

(d) Given that the two dice land on different numbers, find the conditional probability that at least one die is a 6.

Solution:

(a) There are 6 possible outcomes that are doubles, so the probability of doubles is $\frac{6}{36} = \frac{1}{6}$.

(b) The conditioning event (sum is 4 or less) consists of the 6 outcomes

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\},$$

2 of which are doubles, so the conditional probability of doubles is $\frac{2}{6} = \frac{1}{3}$.

(c) There are 11 possible outcomes with at least one 6, namely,

$$(6,6),(6,i),\text{and } (i,6),\text{for } i = 1,2,...,5.$$

The probability that at least one die is a 6 is $\frac{11}{36}$.

(d) There are 30 possible outcomes where the dice land on different numbers. Out of these, there are 10 outcomes in which at least one of the rolls is a 6. Thus, the desired conditional probability is $\frac{10}{30} = \frac{1}{3}$

Ex. 3 Testing for a rare disease. A test for a certain rare disease has 90% accuracy: if a person has the disease, the test results are positive with probability 0.9, and if the person does not have the disease, the test results are negative with probability 0.9. A random person drawn from a certain population has probability 0.001 of having the disease. Given that the person just tested positive, what is the probability of having the disease?

[1] Exercises are taken from chapter 1 of the textbook. Send comments/questions to carlo.ciccarelli@uniroma2.it

Solution: Let A be the event that the person has the disease. Let B be the event that the test results are positive. The desired probability, $P(A|B)$, is found by Bayes rule:

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)} = \frac{0.001 \times 0.90}{0.001 \times 0.90 + 0.999 \times 0.10} = 0.089.$$

Note that even though the test was assumed to be fairly accurate,a person who has tested positive is still very unlikely to have the disease.

Ex.4 A system consists of n identical components that are reliable with probability p independently of other components. The system succeeds if at least k out of the n components work reliably. What is the probability of success ?

Solution: Let A_i be the event that exactly i components work reliably. The probability of success is the probability of the union $\cup_{i=k}^{n} A_i$, and since the A_i are disjoint, it is equal to

$$\sum_{i=k}^{n} P(A_i) = \sum_{i=k}^{n} p(i),$$

where $p(i)$ are the binomial probabilities. Thus, the probability of success is

$$\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$